The Pupil of an Optical System with regard to Perspective.

By Conrad Beck.

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Hitherto the use made of the so-called "entrance and exit pupil" of an optical system has been chiefly in regard to aperture and the questions which are related thereto. I have been led to investigate this conception more completely, and I have now found that it may be used with advantage as an additional factor, in connection with the Gauss planes, for explaining the action of optical instruments as regards the perspective of the images formed. The Gauss planes enable us to refer the action of a complicated optical system to an equivalent single lens placed in two positions—the entrance equivalent plane and the exit equivalent plane. In fig. 1, the upper half of the diagram shows the course of a ray of light from X passing through the

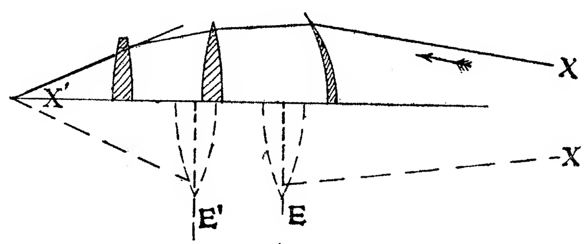


Fig. I

three lenses to X′, each lens bending the light to a certain extent. The lower half of the diagram shows a lens of the correct equivalent focus placed at E to receive the light, and at E′ to discharge it. This will act in a similar manner to the complete system, and many of the properties of the optical system may be correctly studied by confining our attention to an equivalent single lens, placed in the plane E to receive the light, and shifted to the plane E′ to discharge it.

By this means, assuming that the optical system is corrected in such a manner that the oblique rays and those far from the axis act in the same manner as the direct axial rays, the position and size of images can be determined with accuracy; but the perspective of the image cannot be correctly explained by aid of the Gauss planes alone. The "pupils" used as a modifying factor in connection with the Gauss planes will explain this apparent discrepancy.

The conception of the entrance and exit pupils was first emphasised by

Prof. Abbé. The matter has since been discussed in Germany under the title, "the regulation of the rays." The pupils have, however, been treated as though they were in a certain sense antagonistic to the Gauss planes, and their properties do not seem to have been thoroughly worked out.

The entrance and exit pupils are the equivalent apertures of the optical system, and their service is therefore analogous to that of the equivalent planes and equivalent foci of the Gauss system. By assigning the correct position to these two apertures or pupils we can investigate the perspective of an image without taking further consideration of the system itself; just as by assigning the correct position to the two Gauss planes we can investigate the size and position of the images irrespective of the system itself.

The Gauss equivalent lens tells us the size and position of the image formed. The "pupils" determine what rays form the image, and they further determine the perspective, without invalidating the results given by the Gauss system as to the positions and sizes of the focussed images.

Fig. 2 shows a pair of entrance and exit pupils in the simple case of a single lens with a limiting aperture or diaphragm, D, in front of the lens, L.

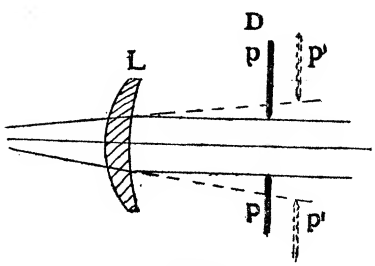


Fig. 2

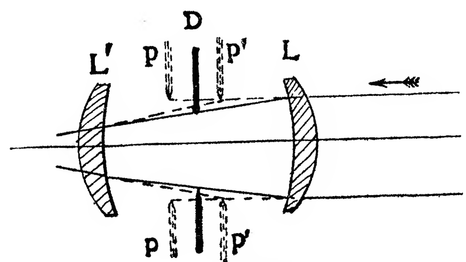


Fig 3

The entrance pupil is, in this case, the actual diaphragm D, but the rays emerge from the lens as if they came from the exit pupil $p'p'$, which is the conjugate image of the stop D seen through the lens L. Whatever may occur to the light before entering or within the optical system, it emerges as coming from the exit pupil $p'p'$.

Fig. 3 shows a pair of lenses, each similar to that in fig. 2, with a limiting aperture D between them.

Here the entrance pupil pp is the conjugate image of the diaphragm D through lens L, and the exit pupil $p'p'$ is, as before, the conjugate image of the diaphragm D through the lens L′. The rays which enter towards the entrance pupils pp emerge as if coming from the exit pupil $p'p'$.

Fig. 4 shows an optical system such as a telephotographic lens, in which the entrance pupil pp and exit pupil $p'p'$ are situated in widely different

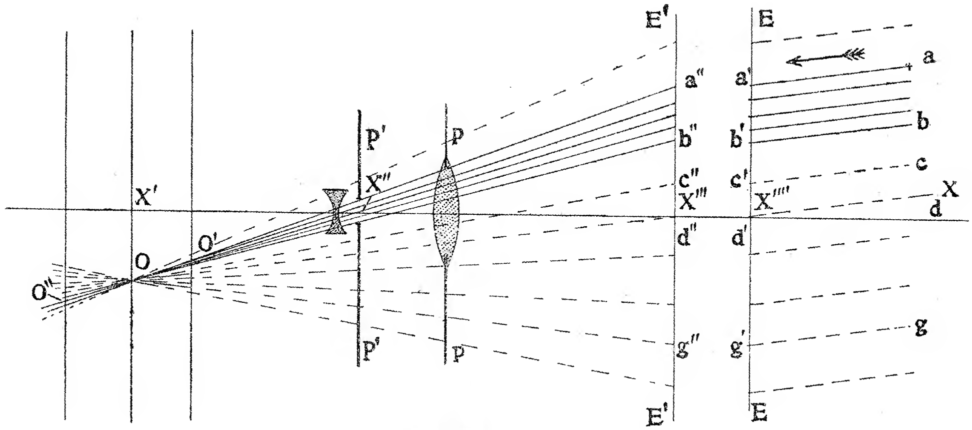


Fig 4

positions from the Gauss equivalent planes EE′. In this case the entrance pupil is situated on the front lens of the system at *pp*, whilst the entrance equivalent plane is at EE, the exit pupil at *p′p′*, whilst the exit equivalent plane is at E′E′.

Fig. 5 shows a lens system of a similar focal length where the exit pupil *p′p′* is in the same position as the exit equivalent plane. A comparison between the two figs. 4 and 5 will demonstrate the effect caused by the exit pupil being displaced in position from that of the equivalent plane.

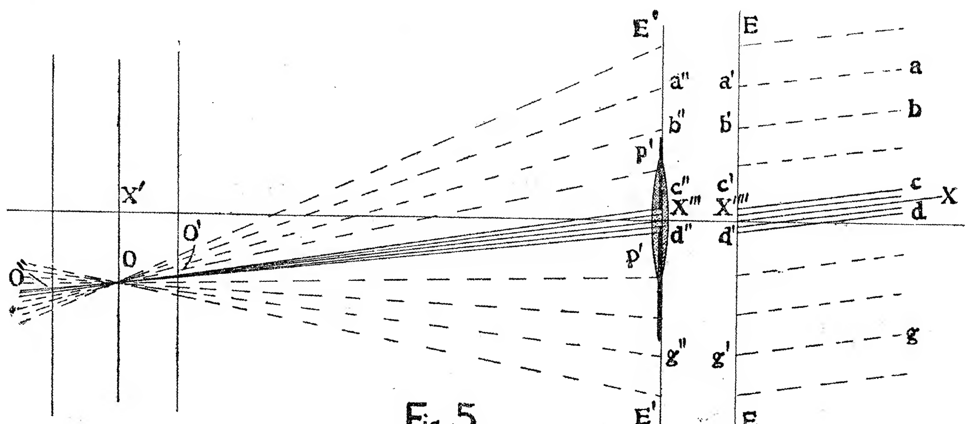


Fig 5

As regards the image of objects to the right of the optical system which are exactly and correctly focussed to any one plane, OX′, to the left of the system, there is no difference—the position and size of the image is the same in both figs. 4 and 5.

One is accustomed to treat optical systems, when using the Gauss planes

for purposes of finding the size and position of images, as though a pinhole might be placed in the point where the back equivalent plane E′E′, fig. 5, cuts the axis at X‴; and lines representing single refracted light rays are drawn from this point into the image-space parallel to incident rays which enter the system towards X⁗. The point X‴ becomes a projection centre for the system, and its position governs the size and position of images. It would appear at first sight that by displacing the exit pupil, as shown in fig. 4, to a distance from the back equivalent plane, this exit pupil X″, fig. 4, becomes the projection centre and the place at which the pinhole should be placed. It has been further stated that this point forms a geometric centre of projection, regulating the perspective, and that the Gauss system cannot be applied to this class of lens. I shall now show that this is not the case.

It may be assumed for argument that in both cases the optical system is large enough, if the limiting apertures be removed, to admit of the large homocentric bundle of rays *abcdg*. These rays in figs. 4 and 5 encounter the entrance equivalent plane at $a'b'c'd'g'$, and are refracted as coming from points $a''b''c''d''g''$ on the exit equivalent plane. The distance $X''''a'$ equals $X'''a''$, and so on. The rays meet at a point O on the conjugate focal plane X′O. The size and position of the image is in both cases (figs. 4 and 5) the same, being regulated only by the equivalent focal length of the lens, and the positions of the Gauss equivalent planes; but in fig. 4 a limiting or selecting aperture, the exit pupil $p'p'$, has been introduced. It selects a small oblique bundle of rays $a''b''$ out of the homocentric bundle $a''g''$ to form the image; while in the case shown in fig. 5, the limiting aperture $p'p'$ being placed in the exit equivalent plane, the central homocentric portion $c''d''$ forms the image.

The exit pupil is a selecting device which, if situated at any position not coinciding with the exit equivalent plane, admits light which is not a homocentric bundle, but which may always be considered as a portion of the large imaginary homocentric bundle which, if the system had been large enough, would have been transmitted. This conception of treating the oblique bundle of rays by which the image is actually formed as though it were a portion of a large homocentric bundle obeying the ordinary laws of the Gauss method of treatment, obviates the erroneous results previously alluded to.

The perspective of a photograph taken with an optical system as depicted in fig. 4 or fig. 5 does not depend upon the qualities of the image which is in theoretically exact focus on the plane X′O. Only one plane in the object-space corresponds to this one exact plane in the image-space, and objects situated in one plane can have no perspective. Perspective deals with the

relative positions of points in the image of objects which are at different distances; in fact with the position of objects which are theoretically out of focus, though they may be depicted sufficiently sharply to be well defined in the image. In order to investigate, therefore, the perspective of an image, we must consider the positions of the small out-of-focus circles of confusion which are the images at the plane X′O, of objects that are not in the same plane as the correctly focussed object. It is here that the difference arises between the image of the two optical systems shown in figs. 4 and 5. In the case of fig. 5 the circles of confusion that will be photographed on the point O will be the images of all objects in the object space which are sharply focussed in the image space at their various distances along the line OX‴. In the case of fig. 4 they will be the images of all objects in the object-space that are sharply focussed in the image-space along the line OX″, and not those focussed along the line OX‴. Thus the out-of-focus objects, situated at different distances, which are photographed at one point on the photograph, are different in the two cases, and the perspective will not be the same.

In the case of fig. 5 the line OX‴ is conjugate to a parallel line X⁗X, and all objects lying in the line X⁗X will be focussed on points lying in the line OX‴. Their circles of confusion will lie symmetrically along this line. The photograph is a true projection from the Gauss planes EE′, and the perspective of the picture is such that if a photograph be taken and viewed by an eye placed at E′, it will give the same perspective rendering as that of the object seen from E.

In the case of a system such as fig. 4, however, the objects which have their images situated on the line OX″ do not all lie on a line in the object-space parallel to OX″, but on another line whose position depends on the position of the exit pupil $p'p'$.

A convenient method of expressing the perspective of a photograph is by defining the distance from the eye at which it should be held in order to give on the retina an image in which the proportions of the parts are in the same perspective as is seen by the eye at the standpoint from which the object was photographed, that is, to define the position of the projection centre of the photograph.

In the case of fig. 5 this distance is evidently E′X′—every straight line through the point X‴ (where the back equivalent plane cuts the axis), which defines the centres of circles of confusion of images which are pictured at any point such as O, is conjugate to a straight line parallel to itself with origin X⁗; and the point X‴ is the projection centre of the photograph corresponding to the projection centre X⁗ of the object, because every line in the image-space which passes through the point X‴ has a conjugate line

parallel to itself which passes through the point X''''. In cases, however, where the exit pupil does not correspond with the equivalent planes, it will be found that the lines on which the circles of confusion lie in the image-space are not parallel to their conjugate lines in the space.

The position of these conjugate lines can be found by reference to fig. 6. The line conjugate to any line OX'', fig. 6, which passes through the centre of

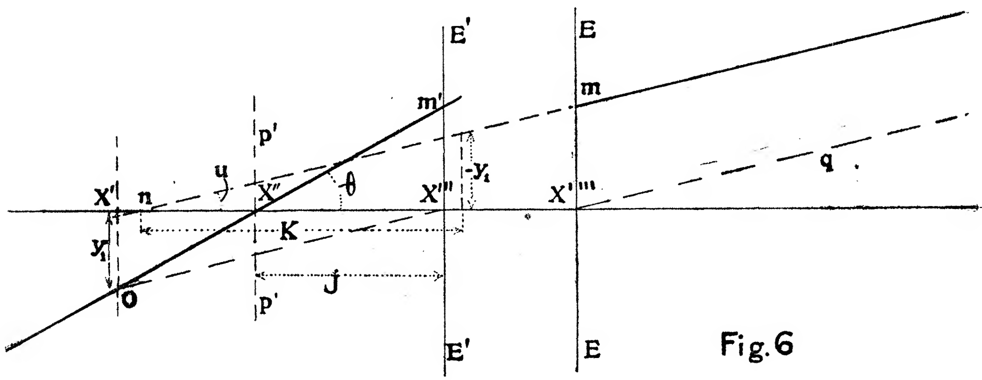


Fig. 6

the exit pupil p' is also a straight line, for let EE and E'E' be the equivalent planes of a system with exit pupil p', take E' as origin for line OX''m', and then call

$$\frac{j}{X'''m'} = \frac{1}{\tan\theta} = b,$$

and the equation for the line OX''m' will be

$$x - by - j = 0. \tag{1}$$

If we insert in this equation for x and y their values in x' and y', their conjugate points, from the ordinary optical formula, E being the origin for co-ordinates y' and x',

$$x = \frac{x'\phi}{\phi - x'}, \qquad y = y'\frac{x}{x'},$$

we get
$$\frac{x'\phi}{\phi - x'} - by'\frac{\phi}{\phi - x'} - j = 0$$

or
$$x'\left(1 + \frac{j}{\phi}\right) - by' - j = 0. \tag{2}$$

This gives the formula for the line nm containing the points conjugate to points on the line OX''m', and it is obviously a straight line. In the case where $y' = 0$

$$x' = \frac{\phi j}{\phi + j}. \tag{3}$$

Therefore the line nm cuts the axis at a point $\phi j/(\phi+j)$ from the equivalent plane E.

Where $x' = 0$,

$$y' = \frac{j}{b} = \mathrm{X}''''m = \mathrm{X}'''m', \tag{4}$$

and it enters the foremost equivalent plane at m, the same distance from the axis as it emerges from the second equivalent plane at m', which is what would be expected.

The distance (K) at which a photograph must be viewed to give the same perspective as that seen from the same standpoint by the eye may be expressed as follows. Let θ be the angle which the line $\mathrm{OX}''m'$ makes with the axis, and let u be the angle which its conjugate line nm makes with the axis. The photograph which has superposed on it at the point O all the images which lie along the line OX″ must be viewed at such a distance that it subtends to the eye the angle u such that the distance $\mathrm{K} = \dfrac{y}{\tan u}$.

Now $$y = (x-j)\tan\theta; \quad \text{thus} \quad \mathrm{K} = (x-j)\frac{\tan\theta}{\tan u},$$

$$\tan\theta = \frac{\mathrm{X}''''m}{j}, \quad \text{and} \quad \tan u = \frac{\mathrm{X}''''m}{n\mathrm{X}''''} = \frac{\mathrm{X}''''m}{x'}.$$

In this case, where $y' = 0$, $x' = \dfrac{\phi j}{\phi+j}$ from (3);

thus $$\frac{\tan\theta}{\tan u} = \frac{\phi}{\phi+j}, \quad \text{and} \quad \mathrm{K} = (x-j)\frac{\phi}{\phi+j}. \tag{5}$$

If $x = -\phi$, $\mathrm{K} = -\phi$. But when x is greater than ϕ numerically, K becomes greater than x.

From these formulæ a very simple geometrical construction can be made for showing the position, in the object-space of an optical system, of a line which is conjugate to any given line in the image-space. Suppose that in fig. 6 X′O is the focal plane of the system; from O draw a line OX‴ to the back equivalent point, and from X⁗ draw a line to $\mathrm{X}''''q$ parallel to OX‴. From a point on the first equivalent plane m, where $\mathrm{X}''''m = \mathrm{X}'''m'$, draw a line parallel to $\mathrm{X}''''q$, and this line will be the conjugate line to $\mathrm{OX}''m'$. All points which lie along the line nm will have their images on the line $\mathrm{OX}''m'$; for from equation (4) where $x' = 0$, $y' = \mathrm{X}''''m$, and K for the focal plane $= \mathrm{OX}'/\tan u$. In this case (the focal plane) $-\mathrm{K} = -\phi$. Thus $-\phi = \mathrm{OX}'/\tan u$; also $-\phi = \mathrm{OX}'/\mathrm{X}'\mathrm{X}'''\mathrm{O}$. Therefore the angle $\mathrm{OX}'''\mathrm{X}' =$ the angle u, which proves that the conjugate line nm is parallel to a line drawn to the equivalent point X‴ from the point O, where the given line OX″M′ cuts the focal plane. The value of K (formula 5) gives

the perspective centre in terms of the focal length, the position of the photographic plate, and the position of the exit pupil. It also shows that for distant objects sharply focussed where $x = -\phi$, $K = -\phi$, and the perspective centre is in the exit Gauss plane at E′, as would be the case when the exit pupil coincides with the exit equivalent plane; but the case is different for near objects, and an anomalous perspective is produced, which is very unexpected.

To take an example: suppose the equivalent focal length of the system (fig. 4) be 9 inches, and the exit pupil is displaced 5·83 inches behind the equivalent plane, then if a photograph of a distant view be taken, the photographic plate being placed in the focal plane, the perspective will be normal for objects nearly at infinity, and exactly like that taken by any other form of 9-inch lens. Hence, to obtain correct perspective, the photograph must be viewed with the eye at a distance away of 9 inches ($K = \phi$), and not, as has been previously suggested, with the eye at the position of the exit pupil, which in the case chosen is at a distance of 3·17 inches. Were the latter the case, no difference in perspective would be gained by the use of a telephoto lens over a short-focus instrument. If a near object is being photographed, and the plate be placed 18 inches away from the exit equivalent plane, so as to photograph objects full size, the result is remarkable, as in this case $K = 35\frac{1}{2}$ inches, and the perspective centre, instead of being 18 inches away, as would be the case with an ordinary lens, or being 12·17 inches away if the exit pupil were the projective centre, is $35\frac{1}{2}$ inches away, and the perspective is greatly reduced by the use of such a lens.

If the object is photographed half-size, the perspective centre is 22 inches, instead of $13\frac{1}{2}$ inches. This explains a very interesting point in the practical use of telephotographic lenses. Such lenses have usually very small apertures and possess a large degree of so-called depth of focus, and are consequently capable of depicting a great range of depth in the object. For distant views the perspective will, on the whole, give the effect produced by photographing with an ordinary lens of about the same focal length as the equivalent focal length of the telephoto system. But if a telephoto lens be used for near objects, as, for instance, for full-size portraits, the perspective of a 9-inch telephoto lens with exit pupil in the position shown in fig. 4 will give the perspective effect produced by an ordinary lens of $17\frac{1}{2}$ inches focus; or, if half full-size, of a lens $14\frac{1}{2}$ inches. This accounts for the very pleasing portraits obtained by the use of the telephoto lens. For all purposes, except extreme distance, the perspective foreshortening in a photograph taken with a telephoto lens is less pronounced than would be expected from a lens of that focal length.

I have taken two photographs with two lenses, one the 9-inch telephoto lens described, with exit pupil displaced to about the extent of the example cited, the other with an ordinary photographic lens (see figs. 7 and 8). Both lenses had the same equivalent focal length. The object photographed was a parallel tube, about 5 feet long, built up of laths covered with tissue-paper. The photograph was taken looking down the tube, and the nearest end was in both cases placed at such a distance as to give the same-sized image of the object. It will be noticed that the telephoto lens gives a totally different and less steep perspective effect than the ordinary lens; in fact, the result is more striking than would have been anticipated.

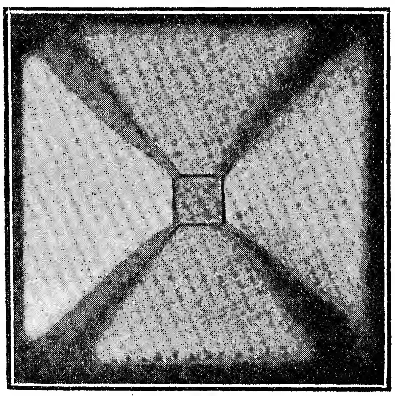


Fig. 7.—Ordinary Lens.

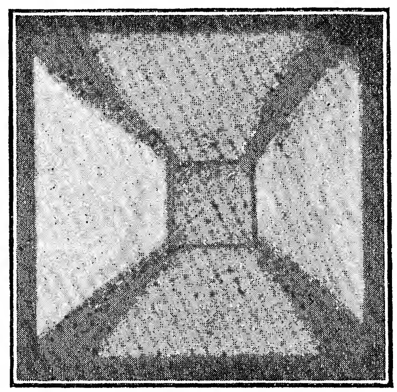
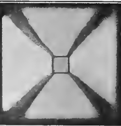


Fig. 8.—Telephoto Lens.

A second pair of photographs, of a row of match-boxes, taken under exactly the same circumstances by the two different types of lenses, shows the same result.



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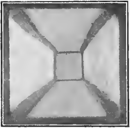

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